1 Quiz 5 (Mar 4) solutions

• 1. Find the general solution and $\lim_{n\to\infty} x_n$ (if it exists). Assume $x_0 = 10$. (a) $x_{n+1} = x_n + 6$, (b) $x_{n+1} = 0.3x_n$, (c) $x_{n+1} - 3x_n = 5$.

(a) $x_{n+1} = x_n + 6$: If a population increases by a fixed number d each time period, we say that the sequence is an arithmetic sequence: $x_{n+1} = x_n + d$. Suppose we know the initial value x_0 ; then the general solution to an arithmetic sequence is $x_n = x_0 + nd$. In this case, $x_0 = 10$, and $d = 6$, so $x_n = 10 + 6n$.

As $n \to \infty$, this gets larger and larger without bound: $x_0 = 10, x_1 = 16, ...x_{100} = 610, ...$ so we say $\lim_{n\to\infty} x_n = +\infty$.

(b) $x_{n+1} = 0.3x_n$: A geometric sequence is defined by $x_{n+1} = rx_n$, where r is a fixed real number. Notice that if we know x_0 , then the general solution to the difference equation represents the x_n in terms of x_0, n, and other given constants. For a geometric sequence, the general solution is $x_n = r^n x_0$.

When $0 < r < 1$, the sequence decays to zero. It has a limit of zero, meaning that as n gets large ($n \to \infty$), $x_n \to 0$, i.e. $\lim_{n\to\infty} x_n = 0$. For $r > 1$, the terms of the sequence increase exponentially.

(c) $x_{n+1} - 3x_n = 5$: Add $3x_n$ to both sides:

$$x_{n+1} = 3x_n + 5$$

The general solution to a difference equation of the form $x_{n+1} = ax_n + b$ where both $a \neq 1$ and $b \neq 0$ is $x_n = (x_0 - \frac{b}{1-a})a^n + \frac{b}{1-a}$. For us, $a = 3$, $b = 5$ $x_0 = 10$, so our solution is $x_n = (10 - \frac{5}{1-3})3^n + \frac{5}{1-3} = \frac{25}{2} * 3^n - \frac{5}{2}$. Note that as $n \to \infty$, $3^n \to \infty$, so $\lim_{n\to\infty} x_n = +\infty$.

- Find the general solution for $x_{n+1} + 2x_n = 4 - x_n$ and $\lim_{n\to\infty} x_n$.

Subtract $2x_n$ from both sides:

$$x_{n+1} = -3x_n + 4$$

Again, the general solution to a difference equation of the form $x_{n+1} = ax_n + b$ where both $a \neq 1$ and $b \neq 0$ is $x_n = (x_0 - \frac{b}{1-a})a^n + \frac{b}{1-a}$. For us, $a = -3$, $b = 4$ $x_0 = 10$, so our solution is $x_n = (10 - \frac{4}{1-(-3)})(-3)^n + \frac{4}{1-(-3)} = 9 * (-3)^n + 1$.

Note that as $n \to \infty$, $(-3)^n$ alternates between positive (when n is even) and negative (when n is odd) (and gets larger in absolute value), so $\lim_{n\to\infty} x_n$ does not exist (it's not going to $+\infty$ like last time).

2 Worksheet (Mar 4) solutions

• (5.11) A level of more than 1500 mg of a certain drug in the body is considered unsafe. Individual doses are 250 mg, and the drug is removed from the body according to the exponential decay equation $x(t) = x_0 e^{-0.1t}$ where t is measured in hours. How frequently can the drug be safely administered?

Remember that if x_n is the amount left in the body when the n'th dose is given, then

$$x_n = \frac{b}{1 - e^{-k\tau}}(1 - e^{-k\tau(1+n)})$$

where b is the dose amount, and τ is the time between doses. Since $(1 - e^{-k\tau(1+n)}) \leq 1$ for all n, $x_{max} = \frac{b}{1-e^{-k\tau}}$.

We just need to find τ so that $x_{max} = \frac{b}{1-e^{-k\tau}} \leq 1500$. Since $b = 250$, and $k = 0.1$ we can use these values to find a τ: if $\frac{250}{1-e^{-0.1\tau}} = 1500$ then, dividing both sides by 1500 and multiplying both sides by

$1 - e^{-0.1\tau}$, $\frac{250}{1500} = 1 - e^{-0.1\tau}$.

Then, subtract 1 from both sides, so $\frac{250}{1500} - 1 = -e^{-0.1\tau}$, and multiply both sides by -1. Now $1 - \frac{250}{1500} = e^{-0.1\tau}$ and we can apply the natural logarithm to both sides: $\ln(1 - \frac{250}{1500}) \approx -0.18 = \ln(e^{-0.1\tau}) = -0.1\tau$ by properties of the natural logarithm ($\log_e$). Thus $\tau = \frac{-0.18}{-0.1} = 1.8$ hours is the shortest amount of time we can wait between doses.

• (5.12) The charge on a nerve cell (neuron) is increased by 1 millivolt every 2 milliseconds. Individual charges decay exponentially according to the formula $x(t) = x_0 e^{-0.05t}$ where t is measured in milliseconds. Thus, if x_0 is the present charge on the cell, the charge remaining after 2 milliseconds is $x(2) = x_0 e^{-0.1}$ millivolts. Let x_n be the charve on the cell after 2n milliseconds. (a) Show that $x_{n+1} = e^{-0.1} x_n + 1$ and solve for x_n if $x_0 = 0$. (b) The "all or nothing" law asserts that the neuron will fire as soon as the total charge on the cell exceeds a certain threshold value. If the neuron fires as soon as the charge exceeds 4 millivolts, how frequently will the neuron fire?

(a) Certainly, after $2n + 2$ milliseconds, the present charge remaining from the 2n'th millisecond is $x_n * e^{-0.1}$ since 2 milliseconds have passed, and since 2 milliseconds have passed, the charge is increased by 1 millivolt. Thus $x_{n+1} = x_n * e^{-0.1} + 1$.

To solve for x_n with $x_0 = 0$, recall that $x_n = (x_0 - \frac{b}{1-a})a^n + \frac{b}{1-a}$. Since for us $a = e^{-0.1}$, $b = 1$, and $x_0 = 0$, $x_n = (0 - \frac{1}{1-e^{-0.1}})(e^{-0.1})^n + \frac{1}{1-e^{-0.1}} \approx -10.5 * (0.9)^n + 10.5$.

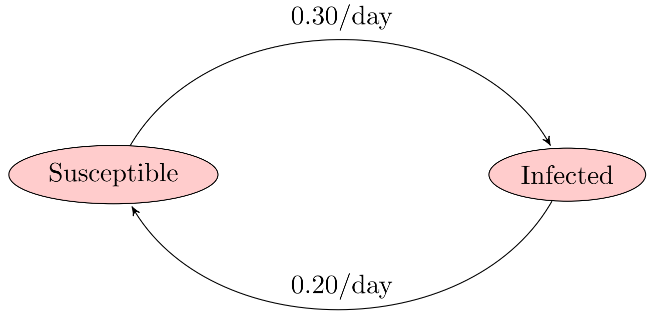
To find out how frequently it will fire, note that the $x_n \approx -10.5 * (0.9)^n + 10.5$ is increasing since $(0.9)^n \to 0$ as $n \to \infty$, so as n gets bigger, the quantity $-10.5 * (0.9)^n$ gets closer to 0 (it gets less negative). Thus once x_n hits 4 millivolts it stays above 4 millivolts.

Let's find when it hits 4: If $x_n = 4$, $-10.5*(0.9)^n + 10.5 = 4$ so $-10.5*(0.9)^n = -6.5$. Then $(0.9)^n = \frac{-6.5}{-10.5}$. Apply ln to both sides: $\ln((0.9)^n) = n * \ln(0.9) = \ln(\frac{-6.5}{-10.5}) \approx -0.48$. Then $n \approx \frac{-0.48}{\ln(0.9)} \approx 4.55$. Thus, for $n = 5$ and onwards, $4 \leq x_n$ and the neuron is constantly firing after $2n = 2 * 5 = 10$ milliseconds.

3 Quiz 6 (Mar 11) solutions

• (i) Suppose that each day, 30% of the dorm residents became infected with a cold, while only 20% of those infected recovered and became susceptible again. Draw the flow diagram for this situation and give the corresponding transfer matrix.

The flow diagram is as follows:



Note that if S(t), I(t) are respectively the amount of students susceptible and infected at the end of day t, then at the end of the next day (day t+1), 30% of the susceptible people from yesterday (there's S(t) of them) got infected. From the people that were sick yesterday (there's I(t) of them), 20% got healthy again.

Thus $S(t+1) = S(t) - (0.3) * S(t) + (0.2) * I(t) = 0.7S(t) + 0.2I(t)$.

Similarly, $I(t+1) = I(t) - (0.2)I(t) + 0.3S(t) = 0.3S(t) + 0.8I(t)$. So our system of equations is

$$S(t+1) = 0.7S(t) + 0.2I(t)$$

$$I(t+1) = 0.3S(t) + 0.8I(t)$$

. Thus, our transfer matrix is

$$\begin{array}{c} \\ S(t+1) \\ I(t+1) \end{array} \begin{array}{c} \begin{matrix} S(t) & I(t) \end{matrix} \\ \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \end{array}$$

Note that the columns sum to 1 and the entries are nonnegative (what kind of matrix is this?)

- (ii) Perform the following multiplications: (a) $\begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and (b) $\begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(a) By the definition of matrix multiplication, $\begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2*0+-1*1 & 2*1+-1*0 \\ 5*0+3*1 & 5*1+3*0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & 5 \end{bmatrix}$

It is helpful to view matrix multiplication as applying the <u>LEFT</u> matrix to each column of the matrix on the <u>RIGHT</u> in general:

$$\begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \left[\begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] = \begin{bmatrix} 2*0+-1*1 & 2*1+-1*0 \\ 5*0+3*1 & 5*1+3*0 \end{bmatrix}$$

(b) Again, by the definition of matrix multiplication, $\begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2*-1+-1*2 \\ 5*-1+3*2 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$.

4 Worksheet (Mar 11) solutions

- 1. Make a matrix equation from the following three linear equations:

$$x_1(t+1) = 0.4x_1(t) + 0.8x_2(t) + 0.1x_3(t)$$

$$x_2(t+1) = 0.9x_1(t)$$

$$x_3(t+1) = 0.7x_2(t)$$

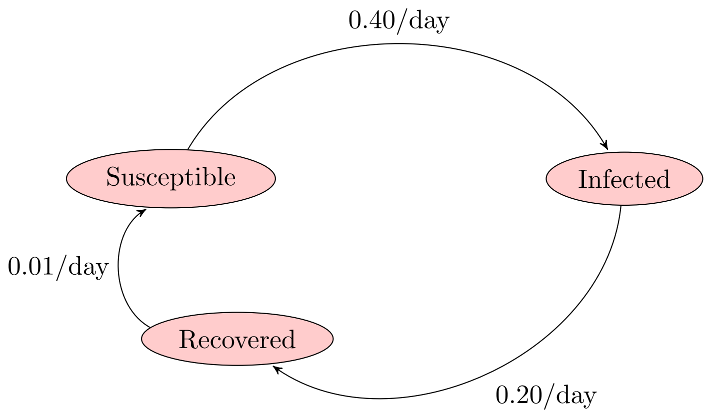
The following matrix models the linear system:

$$\begin{array}{c} \\ x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{array} \begin{array}{c} \begin{matrix} x_1(t) & x_2(t) & x_3(t) \end{matrix} \\ \begin{pmatrix} 0.4 & 0.8 & 0.1 \\ 0.9 & 0 & 0 \\ 0 & 0.7 & 0 \end{pmatrix} \end{array}$$

Thus, our matrix equation would be $\begin{bmatrix} 0.4 & 0.8 & 0.1 \\ 0.9 & 0 & 0 \\ 0 & 0.7 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{bmatrix}$.

- 2.We will divide a population into three classes of people: (1) susceptible, (2) infected, and (3) recovered. Those individuals who recovered are immune temporarily, and then their immunity wanes. A particular dormitory had a flu epidemic. Each day, 40% of the dorm residents became infected, while only 20% of those infected recovered and became immune. Those recovered and temporarily immune students became susceptible at a rate of 1% per day. Draw the flow diagram for this situation and give the corresponding transfer matrix.

The flow diagram is as follows:



Note that if S(t), I(t), and R(t) are respectively the amount of students susceptible, infected, and recovered at the end of day t, then at the end of the next day (day t+1), 40% of the susceptible people from yesterday (there's S(t) of them) got infected. From the people that were recovered and temporarily immune yesterday (there's R(t) of them), 1% became susceptible again. Thus $S(t+1) = S(t) - (0.4) * S(t) + (0.01) * R(t) = 0.6S(t) + 0I(t) + 0.01R(t)$.

Similarly, $I(t+1) = I(t) - (0.2)I(t) + 0.4S(t) = 0.4S(t) + 0.8I(t) + 0R(t)$, and $R(t+1) = R(t) - (0.01)R(t) + 0S(t) + 0.2I(t) = 0S(t) + 0.2I(t) + 0.99R(t)$. So our system of equations is

$$S(t+1) = 0.6S(t) + 0I(t) + 0.01R(t)$$
$$I(t+1) = 0.4S(t) + 0.8I(t) + 0R(t)$$
$$R(t+1) = 0S(t) + 0.2I(t) + 0.99R(t)$$

. Thus, our transfer matrix is

$$\begin{array}{c} \\ S(t+1) \\ I(t+1) \\ R(t+1) \end{array} \begin{array}{c} \begin{array}{ccc} S(t) & I(t) & R(t) \end{array} \\ \begin{pmatrix} 0.6 & 0 & 0.01 \\ 0.4 & 0.8 & 0 \\ 0 & 0.2 & 0.99 \end{pmatrix} \end{array}$$

Note that the columns sum to 1 and the entries are nonnegative (what kind of matrix is this?)

• 3. A basic model for the spread of the herpes simplex virus divides a population into 3 categories (1) susceptible (does not have the virus), (2) infected and shedding the virus (can infect others), and (3) quiescent (infected but not shedding the virus), and the remaining proportion of the population is in the quiescent phase. Construct a vector that represents what proportion of the population is in each category.

If we label the amount of people that are susceptible by S, infected by I, and quiescent by Q, the proportion of the population of susceptible people (for example) is $\frac{S}{S+I+Q}$ since $S+I+Q$ is the how many total people there are in the population (since everyone falls into one of the categories). Then $\begin{bmatrix} \frac{S}{S+I+Q} \\ \frac{I}{S+I+Q} \\ \frac{Q}{S+I+Q} \end{bmatrix}$ is a vector representing what proportion of the population is in each category.